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Corner Transfer Matrices of the Eight-Vertex Model. II. The Ising Model Case

R. J. Baxter[1]

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In a previous paper certain "corner transfer matrices" were defined. It was conjectured that for the zero-field, eight-vertex model these matrices have a very simple eigenvalue spectrum. In this paper these conjectures are verified for the case when the eight-vertex model reduces to two independent and identical square-lattice Ising models. The Onsager–Yang expression for the magnetization follows immediately.

KEY WORDS: Statistical mechanics; lattice statistics; eight-vertex model; Ising model; corner transfer matrices; spontaneous magnetization.

1. INTRODUCTION

In a previous paper,[1] hereafter referred to as I, two "corner transfer matrices" (CTMs) A and B were defined for the zero-field, eight-vertex model. It was conjectured (in the limit of an infinite lattice) that A and B commute, that both are exponentials of simple Heisenberg-like operators, and that their diagonal forms are simple direct products.

For the case when the eight-vertex model reduces to two independent and identical square-lattice Ising models, the matrices A and B can be handled by Kaufman's[2] spinor representation. This is done in this paper, and the conjectures are verified for this case. The Onsager–Yang[3,4] expression for the magnetization of the Ising model follows immediately.

In Sections 5 and 6 the temperature T is taken to be less than the critical value T_c. The case $T > T_c$ is briefly discussed in Section 7.

[1] Research School of Physical Sciences, The Australian National University, Canberra, Australia.

2. DEFINITION OF THE CTM

Consider a typical face of the square lattice, as shown in Fig. 1, with spins $\sigma_1, \sigma_2, \sigma_{2'}, \sigma_3$ (equal to $+1$ or -1) at its corners. The contribution of this face to the energy is

$$\epsilon = -J\sigma_2\sigma_{2'} - J'\sigma_1\sigma_3 - J_4\sigma_1\sigma_2\sigma_{2'}\sigma_3 \tag{1}$$

(using the spin formulation of the eight-vertex model[5,6]). Going downward and to the right, the operator V_1 that adds this face to the lattice is one with elements

$$(V_1)_{\sigma_1\sigma_2\sigma_3|\sigma_1'\sigma_2'\sigma_3'} = \delta_{\sigma_1\sigma_1'}\,\delta_{\sigma_3\sigma_3'}e^{-\beta\epsilon} \tag{2}$$

Define the Pauli operators acting on spin j to be

$$s_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad c_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad d_j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3}$$

Then after a little algebra one can verify that

$$\begin{aligned} V_1 &= \tfrac{1}{2}[a + d + (a - d)s_1s_3 + (b + c)c_2 + (c - b)s_1c_2s_3] \\ &= \tau \exp(Lc_2 + L's_1s_3 + L''s_1c_2s_3) \end{aligned} \tag{4}$$

where

$$\begin{aligned} a &= \exp[\beta(J + J' + J_4)] = \tau(\exp L')\cosh(L + L'') \\ b &= \exp[\beta(-J - J' + J_4)] = \tau \exp(-L')\sinh(L - L'') \\ c &= \exp[\beta(-J + J' - J_4)] = \tau(\exp L')\sinh(L + L'') \\ d &= \exp[\beta(J - J' - J_4)] = \tau \exp(-L')\cosh(L - L'') \end{aligned} \tag{5}$$

These a, b, c, and d are the usual Boltzmann weights of the eight-vertex model.[7]

Now consider the lattice of 24 squares shown in Fig. 2. Label the spins outward from the center as indicated. Then the operator that adds the upper right corner to the lattice (going clockwise) is

$$A = V_3V_2V_1 \cdot V_3V_2 \cdot V_3 \tag{6}$$

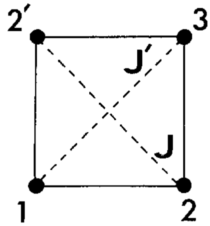

Fig. 1. Typical face of the square lattice, surrounded by sites 1, 2, 2′, 3. The broken lines represent the two-spin interactions between spins on a diagonal.

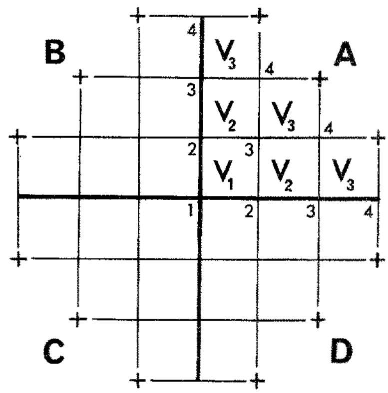

Fig. 2. The square lattice for the case $n = 3$, showing the division into four quadrants corresponding to the corner transfer matrices A, B, C, and D. Also shown are the single-face operators V_1, V_2, and V_3 that occur in Eq. (6) for A.

where V_2 (V_3) is defined similarly to V_1, but acts on spins 2, 3, 4 (3, 4, 5). Thus in general

$$V_j = \tfrac{1}{2}[a + d + (a - d)s_j s_{j+2} + (b + c)c_{j+1} + (c - b)s_j c_{j+1} s_{j+2}] \tag{7}$$

which is Eq. (18a) of I.

The lattice of Fig. 2 can be generalized in an obvious way to one with spins $1, \ldots, n + 1$ on the right horizontal and upper vertical axes. It then has $2n(n + 1)$ squares, and the upper right "corner transfer matrix" (CTM) is

$$A = G_1 G_2 G_3 \cdots G_n \tag{8a}$$

where

$$G_j = V_n V_{n-1} V_{n-2} \cdots V_j \tag{8b}$$

Similar CTMs B, C, and D can be defined for the upper left, bottom left, and bottom right quadrants of the lattice. It is easy to see that

$$C = A, \qquad D = B \tag{9}$$

and that B is obtained from A by interchanging J and J'. The partition function is then

$$Z = \mathrm{Tr}(ABCD) = \mathrm{Tr}(AB)^2 \tag{10}$$

It is convenient to impose the boundary condition that the outermost spins all be positive, as indicated in Fig. 2, which is equivalent to setting $s_{n+2} = 1$ in Eqs. (7) and (8). The magnetization can then be defined as the average value of the center spin σ_1, i.e.,

$$M = \mathrm{Tr}[S(AB)^2]/\mathrm{Tr}(AB)^2 \tag{11}$$

where

$$S = s_1 \tag{12}$$

is the operator which is $+1$ for $\sigma_1 = +1$, and -1 for $\sigma_1 = -1$. Since spin 1 enters Eqs. (7) and (8) only via s_1, S commutes with A and B.

The operators S and AB can therefore be simultaneously diagonalized, thereby simplifying (10) and (11). This is the object of this series of papers.

Arrow Formulation. It is convenient to go from a spin to an arrow formulation, by replacing $\sigma_1,..., \sigma_{n+1}$ by $\mu_1,..., \mu_{n+1}$, where

$$\mu_j = \sigma_j\sigma_{j+1}, \qquad j = 1,..., n + 1 \tag{13}$$

and again we take $\sigma_{n+2} = 1$. The definitions (4), (7), and (12) are then replaced by

$$\begin{aligned} V_j &= \tfrac{1}{2}[a + d + (a - d)s_js_{j+1} + (b + c)c_jc_{j+1} + (b - c)d_jd_{j+1}] \\ &= \tau \exp(Lc_jc_{j+1} + L's_js_{j+1} - L''d_jd_{j+1}), \qquad j = 1,..., n \end{aligned} \tag{14}$$

$$S = s_1s_2 \cdots s_{n+1} \tag{15}$$

This is the arrow formulation (18b), (19b) of I, with a minor change of the boundary condition. The spin formulation will not be used again in this paper, so A is now to be regarded as defined by (8) and (14).

3. KAUFMAN REPRESENTATION OF OPERATORS

The equations of Section 2 are valid for the general eight-vertex model. From now on we consider only the case

$$J_4 = 0 \tag{16}$$

when the model reduces to two noninteracting Ising models on the two sublattices. From (5), it follows that in this case

$$ab = cd, \qquad \tanh L = e^{-2\beta J}, \qquad L' = \beta J', \qquad L'' = 0 \tag{17}$$

The great advantage of this case is well known from the work of Onsager[8] and Kaufman[2]: The 2^{n+1} by 2^{n+1} matrices V_j, G_j, and A are members of a group $\mathscr{G}$, and any matrix in this group can be represented by a $2n + 2$ by $2n + 2$ matrix (which is a vast simplification).

To see this, define a set of anticommuting operators $\Gamma_1,..., \Gamma_{2n+2}$ by

$$\begin{aligned} \Gamma_1 &= s_1, \qquad \Gamma_2 = c_1 \\ \Gamma_{2j-1} &= d_1d_2 \cdots d_{j-1}s_j \\ \Gamma_{2j} &= d_1d_2 \cdots d_{j-1}c_j, \qquad j = 1,..., n + 1 \end{aligned} \tag{18}$$

One can then verify that (for $j = 1,\ldots, n$)

$$
\begin{aligned}
V_j\Gamma_l V_j^{-1} &= \gamma\Gamma_{2j-1} - i\delta\Gamma_{2j+2} && \text{if} \quad l = 2j-1 \\
&= \gamma'\Gamma_{2j} + i\delta'\Gamma_{2j+1} && \text{if} \quad l = 2j \\
&= -i\delta'\Gamma_{2j} + \gamma'\Gamma_{2j+1} && \text{if} \quad l = 2j+1 \\
&= i\delta\Gamma_{2j-1} + \gamma\Gamma_{2j+2} && \text{if} \quad l = 2j+2 \\
&= \Gamma_l \quad \text{for all other values of } l
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\gamma &= \coth(2\beta J), & \delta &= \operatorname{cosech}(2\beta J) \\
\gamma' &= \cosh(2\beta J'), & \delta' &= \sinh(2\beta J')
\end{aligned}
\tag{20}
$$

Thus each operator V_j belongs to the set of operators X such that

$$
X\Gamma_l X^{-1} = \sum_{k=1}^{2n+2} x_{k,l}\Gamma_k, \qquad l = 1,\ldots, 2n+2 \tag{21}
$$

where the $x_{k,l}$ are scalar coefficients.

Let $\hat{X}$ be the $2n + 2$ by $2n + 2$ matrix with elements $x_{k,l}$. We call $\hat{X}$ the *representative* of X. Then it is easy to see that:

(a) The operators X form a group $\mathscr{G}$.

(b) If X, Y, Z are members of $\mathscr{G}$ such that

$$XY = Z \tag{22a}$$

then

$$\hat{X}\hat{Y} = \hat{Z} \tag{22b}$$

(c) If X, Y are members of $\mathscr{G}$ such that

$$\hat{X} = \hat{Y} \tag{23a}$$

then

$$X = \lambda Y \tag{23b}$$

where λ is a scalar factor. This means that a representative matrix determines its parent operator to within a scalar factor.

Let $\mathscr{D}$ be the $2n + 2$ by $2n + 2$ diagonal matrix with elements

$$
\begin{aligned}
\mathscr{D}_{j,j} &= +1 && \text{if} \quad j = 1 \text{ or } 2 \text{ (modulo 4)} \\
&= -1 && \text{if} \quad j = 3 \text{ or } 4 \text{ (modulo 4)}
\end{aligned}
\tag{24}
$$

Then from (3) and (18),

$$\Gamma_j^* = \Gamma_j^T = \mathscr{D}_{j,j}\Gamma_j \tag{25}$$

Using these relations, together with the fact that $\Gamma_1,\ldots,\Gamma_{2n+2}$ anti-commute, one can verify the following properties of representative matrices:

(d) Any representative $\hat{X}$ is an orthogonal matrix.

(e) If X is real and symmetric, then $\hat{X}$ is Hermitian and

$$\hat{X}^* = \mathscr{D}\hat{X}\mathscr{D} \tag{26}$$

(f) If X is real and orthogonal, then $\hat{X}$ is real and

$$\hat{X} = \mathscr{D}\hat{X}\mathscr{D} \tag{27}$$

(g) If X is diagonal, then $\hat{X}$ is block-diagonal in the form

$$\hat{X} = \begin{pmatrix} \lambda_0 & & & & & & \\ & \lambda_1 & & & & 0 & \\ & & \lambda_2 & & & & \\ & 0 & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \lambda_n & \\ & & & & & & \lambda_{n+1} \end{pmatrix} \tag{28}$$

where λ_0 and λ_{n+1} are one by one matrices and $\lambda_1,\ldots,\lambda_n$ are two by two orthogonal matrices. (This follows from the fact that $\hat{X}$ must commute with $\hat{s}_1, \hat{s}_2,\ldots, \hat{s}_{n+1}$.)

Since $V_1,\ldots, V_n$ belong to the group $\mathscr{G}$, all our calculations can now be performed on representatives, instead of the original operators. It is convenient to break the representative matrices up into two by two blocks; thus

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n+1} \\ \vdots & & & & \\ a_{n+1,1} & & \cdots & & a_{n+1,n+1} \end{pmatrix} \tag{29}$$

where each a_{ij} is a two by two matrix. $\hat{A}$ can be evaluated by using (19) and (21) to write down $\hat{V}_j$, then putting carets on the matrices in (8) and solving recursively for $\hat{A}$.

For given i and j, one finds that a_{ij} is independent of n, provided n is sufficiently large. It appears that in the infinite-lattice limit one is justified in treating $\hat{A}$ as an infinite-dimensional matrix, with elements equal to the large-n limits of a_{ij}. We shall do this from now on.

The limiting values of the a_{ij} can be calculated by introducing a generating function

$$\mathscr{A}(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x^{i-1}y^{j-1}a_{ij} \tag{30}$$

where each a_{ij} on the rhs is given its limiting, large-n value. After some algebra, one obtains

$$\mathscr{A}(x, y) = \Delta^{-1}\begin{pmatrix} \gamma - \gamma' xy & i(\gamma\delta' x + \delta\gamma' y) \\ -i(\delta\gamma' x + \gamma\delta' y) & \gamma' - \gamma xy \end{pmatrix} \tag{31}$$

where

$$\Delta = 1 - \delta\delta'(x^2 + y^2) - 2\gamma\gamma' xy + x^2 y^2 \tag{32}$$

Since A is symmetric, from property (e) we should have (for x and y real)

$$\mathscr{A}(x, y) = \mathscr{A}^{\dagger}(y, x) = \mathscr{A}^{*}(-x, -y) \tag{33}$$

and indeed these conditions are satisfied.

4. DIAGONALIZATION OF A

Our aim is to diagonalize the product AB, but as a first step we seek to diagonalize A, i.e., to find an operator P such that

$$AP = PA_d \tag{34}$$

where A_d is diagonal.

In fact, we shall find that the same operator P diagonalizes both A and B, and hence AB.

Take representatives of (34), i.e., put carets on each matrix. From (28), $\hat{A}_d$ must be of the form

$$\hat{A}_d = \begin{pmatrix} \lambda_0 & & & & \\ & \lambda_1 & & 0 & \\ & & \lambda_2 & & \\ & 0 & & \lambda_3 & \\ & & & & \ddots \end{pmatrix} \tag{35}$$

where λ_0 is a one by one orthogonal matrix and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are two by two orthogonal matrices. It is then natural to write

$$\hat{P} = \begin{pmatrix} p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ p_{30} & p_{31} & p_{32} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{36}$$

where the p_{i0} are two by one matrices and the other p_{ij} are two by two. The representative equation of (34) can then be written

$$\sum_{k=1}^{\infty} a_{ik} p_{kj} = p_{ij} \lambda_j \tag{37}$$

where $i = 1, 2, 3, \ldots$ and $j = 0, 1, 2, \ldots$.

To handle this equation, we introduce a generating function for column j of $\hat{P}$:

$$p_j(x) = \sum_{i=1}^{\infty} x^{i-1} p_{ij} \tag{38}$$

From (30) and (37) it then follows that, for x in some neighborhood of the origin,

$$(1/2\pi i) \int_C \mathscr{A}(x, y^{-1}) p_j(y) y^{-1} \, dy = p_j(x) \lambda_j \tag{39}$$

where C is a simple closed contour in the y plane, surrounding the origin and such that $p_j(y)$ is analytic inside and on C, while $\mathscr{A}(x, y^{-1})$ is analytic outside and on C.

5. REDUCTION TO A DIFFERENCE KERNEL

Equation (39) is an integral equation for $p_j(x)$ and the eigenvalue matrix λ_j. It is remarkable that it can be solved by a change of variables that reduces it to an integral equation with a difference kernel. Such transformations occur in all the exactly solvable nearest neighbor two-dimensional models (see, for example, p. 11 of Ref. 3, p. 169 of Ref. 9, p. 3120 of Ref. 10). They usually involve introducing elliptic functions.

First note from (39), (30)–(32), and (20) that the kernel of the integral equation has a denominator proportional to

$$\begin{aligned} \Delta_1 = {} & (\cosh 2\beta J)(\cosh 2\beta J') - \frac{1}{2}(\sinh 2\beta J)\left(\frac{x}{y} + \frac{y}{x}\right) \\ & + \frac{1}{2}(\sinh 2\beta J')\left(xy + \frac{1}{xy}\right) \end{aligned} \tag{40}$$

If we set

$$x/y = e^{i\omega}, \qquad xy = -e^{i\omega'} \tag{41}$$

then Δ_1 is an expression that occurs in the standard methods of solving the Ising model, for example, in Eq. (108) of Onsager.[8]

Now consider temperatures less than the critical temperature, and use in (20) the elliptic function parametrization of Onsager [Eqs. (2.1) and (2.2) of Ref. 8]:

$$\begin{aligned} k &= (\sinh 2\beta J \sinh 2\beta J')^{-1}, && 0 < k < 1 \\ \operatorname{am}(ia) &= 2i\beta J', && 0 < a < K' \\ \gamma &= \operatorname{dn} ia, \qquad \delta = -ik \operatorname{sn} ia \\ \gamma' &= \operatorname{cn} ia, \qquad \delta' = -i \operatorname{sn} ia \end{aligned} \tag{42}$$

Here k is the modulus of the elliptic functions and $2K$ and $2iK'$ are the periods. Hereafter the parameter a is that defined by (42), and is not to be confused with the Boltzmann weight a defined by (5).

Change variables from x, y to u, v, where

$$x = -ik^{1/2} \operatorname{sn} u, \qquad y = -ik^{1/2} \operatorname{sn} v \tag{43}$$

and let

$$p_j'(u) = p_j(-ik^{1/2} \operatorname{sn} u) \tag{44}$$

Making these substitutions into (39), we find that the equation becomes

$$(1/2\pi i) \int_{C_1} W(u, v) p_j'(v)\, dv = p_j'(u) \lambda_j \tag{45}$$

where C_1 is the mapping of C into the v plane and

$$W(u, v) = \mathscr{A}(x, y^{-1})\, d(\ln y)/dv \tag{46}$$

Using (42), (43), and (30)–(32), after some considerable algebra, one finds that

$$W(u, v) = L^{-1}(u) M(u, v) L(v) \tag{47}$$

where

$$L(u) = \begin{pmatrix} \operatorname{dn} u & -ik^{1/2} \operatorname{cn} u \\ \operatorname{dn} u & ik^{1/2} \operatorname{cn} u \end{pmatrix} \tag{48}$$

$$M(u,v) = \begin{pmatrix} \phi(v - u + ia) & \phi(v + u + 2K - ia) \\ \phi(v + u + 2K + ia) & \phi(v - u - ia) \end{pmatrix} \tag{49}$$

and the function $\phi(u)$ is given by

$$\phi(u) = (\operatorname{cn} u + \operatorname{dn} u)/(2 \operatorname{sn} u) \tag{50}$$

Substituting the expression (46) for $W(u, v)$ into (45) and defining

$$f_j(u) = L(u) p_j'(u) \tag{51}$$

the integral equation becomes

$$(1/2\pi i) \int_{C_1} M(u, v) f_j(v)\, dv = f_j(u) \lambda_j \tag{52}$$

From (49), the elements of the kernel $M(u, v)$ are functions only of either $u - v$ or $u + v$. By negating u and v in the lower element(s) of $f_j(u)$ and $f_j(v)$, we can ensure that the kernel is a function only of $u - v$, i.e., it is a difference kernel.

We still need to prescribe the contour C_1. To do this, note that under the transformation (43), and for $0 < \zeta < K'$, the line element $(i\zeta - 2K, i\zeta + 2K)$ in the v plane maps to a simple closed curve C surrounding the origin in the y plane. As v moves from left to right along this element, y moves *clockwise* once around C.

Thus we can take C_1 to be this line element, provided we negate the lhs of (52) and choose ζ so that C surrounds the singularities of $\mathscr{A}(x, y^{-1})$. These singularities correspond to the poles of $M(u, v)$, and an appropriate choice of ζ is one such that

$$a + |\mathrm{Im}(u)| < \zeta < 2K' - a - |\mathrm{Im}(u)| \tag{53}$$

With this choice, and when v lies on C_1, the imaginary part of the argument of each function ϕ in (49) lies in the interval $(0, 2K')$. Thus ϕ there can be replaced by its Fourier series

$$\phi(u) = -\frac{\pi i}{2K} \sum_{m=-\infty}^{\infty} \frac{1}{1 + q^m} e^{im\pi u/2K} \tag{54}$$

where

$$q = e^{-\pi K'/K} \tag{55}$$

is the nome of the elliptic functions. The series (54) is convergent in the strip $0 < \mathrm{Im}(u) — 2K'$.

The integral equation (52) can now be solved by Fourier series. Remembering that λ_0 is a one by one matrix and $\lambda_1, \lambda_2, \ldots$ are orthogonal two by two matrices, we obtain

$$\lambda_0 = 1, \qquad f_0(u) = \begin{pmatrix} g_0 \\ g_0 \end{pmatrix} \tag{56a}$$

and, for $m = 1, 2, 3, \ldots$,

$$\lambda_m = \begin{pmatrix} \cosh(\pi ma/2K) & -i\sinh(\pi ma/2K) \\ i\sinh(\pi ma/2K) & \cosh(\pi ma/2K) \end{pmatrix} \tag{56b}$$

$$f_m(u) = e^{-i\pi mu/2K}\begin{pmatrix} -ih_m & h_m \\ ig_m & g_m \end{pmatrix} + (-1)^m e^{i\pi mu/2K}\begin{pmatrix} ig_m & g_m \\ -ih_m & h_m \end{pmatrix} \tag{56c}$$

where $|\mathrm{Im}(u)| < K'$ and $g_0, g_1, \ldots, h_1, h_2, \ldots$ are constants, as yet arbitrary.

The requirement that $\hat{P}$ be orthogonal implies that

$$g_0^2 = \pi/2K, \qquad g_m h_m = \pi q^{m/2}/[2K(1 + q^m)], \qquad m = 1, 2, \ldots \tag{57}$$

while, using (27), the requirement that $\hat{P}$ be the representative of an orthogonal operator implies that

$$h_m = g_m \tag{58}$$

These conditions determine the g_m and h_m (to within choices of sign).

From (56), each $f_m(u)$ is an entire function. From (48) and (51), $p_m'(u)$ is meromorphic, with possible simple poles when cn u dn u vanishes, i.e., at

$$u = (2m + 1)K + inK' \tag{59}$$

(m and n integers). From (56) and (58) the singularities at $u = (2m + 1)K$ are removable, so each $p_m'(u)$ is analytic in the strip

$$|\mathrm{Im}(u)| < K' \tag{60}$$

One can also verify from (48), (51), and (56) that

$$p_m'(2K - u) = p_m'(u) \tag{61}$$

From (44) it then follows that $p_m(x)$ is analytic in a plane with cuts along the imaginary axis from $ik^{-1/2}$ to $+i\infty$ and from $-ik^{-1/2}$ to $-i\infty$. In particular, it is analytic for

$$|x| < k^{-1/2}$$

and is analytic inside the contour C obtained by mapping the line segment C_1 back into the x plane. Thus we have obtained the required solutions of the original integral equation (39).

6. DIAGONAL FORM OF THE CTM

The representative of A_d is now given by (35) and (56). Comparing representatives and using the property (23), it follows that

$$A_d = P^{-1}AP = \rho \exp[(\pi a/4K)(s_1 s_2 + 2s_2 s_3 + 3s_3 s_4 + 4s_4 s_5 + \cdots)] \tag{62}$$

where ρ is an unknown scalar factor.

One can verify that $\hat{S}$ commutes with $\hat{P}$, and hence S with P, and so, from (17),

$$S_d = P^{-1}SP = S = s_1 s_2 s_3 \cdots \tag{63}$$

The other CTM, namely B, is obtained from A by interchanging J and J'. From (42), this leaves k unchanged but replaces a by

$$a' = K' - a \tag{64}$$

However, from (56)–(58), the functions $f_m(u)$ are *independent of a*. From (51), (48), and (44), the same is true of each $p_m(x)$. Thus $\hat{P}$ and P are independent of a.

It follows that the same operator P diagonalizes both A and B, that A and B commute (in the large-lattice limit), and

$$B_d = P^{-1}BP = \rho' \exp[(\pi a'/4K)(s_1s_2 + 2s_2s_3 + 3s_3s_4 + 4s_4s_5 + \cdots)] \quad (65)$$

where ρ' is another unknown scalar factor.

From (62), it also follows that

$$A = \rho e^{a\mathscr{H}} \quad (66)$$

where the operator $\mathscr{H}$ is defined by

$$\mathscr{H} = (\pi/4K)P\left\{\sum_{j=1}^{\infty} js_js_{j+1}\right\}P^{-1} \quad (67)$$

and is independent of a. One can evaluate $\mathscr{H}$ directly by expanding A to first order in a. From (17), (20), and (42)

$$2L = ka + \mathscr{O}(a^3), \qquad 2L' = a + \mathscr{O}(a^3), \qquad L'' = \mathscr{O} \quad (68)$$

Substituting these expressions into (16), (9), and (8), one obtains a result of the form (66), with

$$\mathscr{H} = \lambda e + \tfrac{1}{2}\sum_{j=1}^{\infty} j(s_js_{j+1} + kc_jc_{j+1}) \quad (69)$$

e being the identity operator and λ some unknown scalar constant. Thus $\ln A$ is an XY operator with coefficients proportional to j.

One can obtain $\hat{P}$ from (67) and (69), the working being rather simpler than that required for diagonalizing A. The same result is of course obtained.

Returning to Eqs. (62)–(65), these can be put into a simpler form by changing from the arrow representation to the third representation of paper I, i.e., by changing from arrow spins $\mu_1, \mu_2,\ldots$ to "v-spins" $v_1, v_2,\ldots$, where

$$v_j = \mu_j\mu_{j+1}, \qquad j = 1, 2,\ldots \quad (70)$$

In this representation Eqs. (62), (65), and (63) become

$$\begin{aligned} A_d &= \rho \exp[(\pi a/4K)(s_1 + 2s_2 + 3s_3 + \cdots)] \\ B_d &= \rho' \exp[(\pi a'/4K)(s_1 + 2s_2 + 3s_3 + \cdots)] \\ S &= s_1s_3s_5s_7 \cdots \end{aligned} \quad (71)$$

To within multiplicative scalar factors for A_d and B_d, these equations can be written as

$$\begin{aligned} A_d &= \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w^3 \end{pmatrix} \otimes \cdots \\ B_d &= \begin{pmatrix} 1 & 0 \\ 0 & w' \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w'^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w'^3 \end{pmatrix} \otimes \cdots \\ S &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \end{aligned} \tag{72}$$

where, using (55) and (64),

$$w = e^{-\pi a/2K}, \qquad w' = e^{-\pi a'/2K} = q^{1/2}/w \tag{73}$$

Thus the corner transfer matrices A and B may be simultaneously diagonalized and written in the simple direct product form (72).

The spontaneous magnetization can now be trivially calculated by using the diagonal forms (72) in (11) (the unknown scalar factors in A_d and B_d cancel out). Using (73), this gives

$$M = \frac{1-q}{1+q}\,\frac{1-q^3}{1+q^3}\,\frac{1-q^5}{1+q^5}\cdots \tag{74}$$

Using standard elliptic function identities, it follows that

$$\begin{aligned} M &= k'^{1/4} = (1-k^2)^{1/8} \\ &= (1 - \operatorname{cosech}^2 2\beta J \operatorname{cosech}^2 2\beta J')^{1/8} \end{aligned} \tag{75}$$

which is the well-known Onsager–Yang result.[3,4]

7. THE CASE $T > T_c$

The results of Sections 5 and 6 are valid only for $k < 1$, i.e., for temperatures less than the critical temperature T_c. If $k > 1$, then from the last paragraph of Section 5, $p_m(x)$ has singularities inside the unit circle, which means that the column vectors of the infinite-dimensional orthogonal matrix $\hat{P}$ cannot be normalized. The equations as written are therefore meaningless for $T > T_c$.

However, one can study the high-temperature case by using duality, i.e., by interchanging each s_j and c_j in (14) and (15). The only effect on the calculation of A_d and B_d is to interchange L and L'. Thus for $T > T_c$ the CTMs still reduce to the diagonal form given by (72) and (73), but now k and a are defined by

$$\begin{aligned} k &= \sinh 2\beta J \sinh 2\beta J', && 0 < k < 1 \\ \operatorname{am}(ia) &= 2i\beta J^* = i \ln \coth \beta J, && 0 < a < K' \end{aligned} \tag{76}$$

With these definitions, A_d and B_d are given in the arrow representation by (62) and (65), but S is now given by

$$S = c_1 c_2 c_3 \cdots \tag{77}$$

From (11) it follows that $M = 0$ for $T > T_c$, as expected.

8. CONCLUSIONS

In paper I it was conjectured for the general eight-vertex model that the corner transfer matrices A and B commute, that both are exponentials of Heisenberg-type operators, and that their diagonal forms are simple direct products. These conjectures have here been verified (subject to a rather cavalier treatment of infinite-dimensional matrices) for the special case when the eight-vertex model reduces to two independent and identical Ising models. This gives a new derivation of the spontaneous magnetization (but not of the free energy).

It should be noted that the elliptic functions used in paper I have a different modulus k than those used in this paper (paper II). Adding subscripts I and II, respectively, to the parameters of the two papers, the correspondence is (for $T < T_c$)

$$\begin{gathered} k_{\mathrm{II}} = 2k_{\mathrm{I}}^{1/2}/(1 + k_{\mathrm{I}}), \qquad K_{\mathrm{II}} = (1 + k_{\mathrm{I}})K_{\mathrm{I}}, \qquad K'_{\mathrm{II}} = \tfrac{1}{2}(1 + k_{\mathrm{I}})K_{\mathrm{I}}' \\ \eta_{\mathrm{I}} = iK_{\mathrm{I}}'/4, \qquad q_{\mathrm{II}} = x_{\mathrm{I}}^2 = q_{\mathrm{I}}^{1/2}, \qquad ia_{\mathrm{II}} = (1 + k_{\mathrm{I}})(\eta_{\mathrm{I}} + v_{\mathrm{I}}) \\ w_{\mathrm{II}} = w_{\mathrm{I}}, \qquad w_{\mathrm{II}}' = w_{\mathrm{I}}', \qquad \tfrac{1}{2}a_{\mathrm{II}}k_{\mathrm{II}} = \xi_{\mathrm{I}} \end{gathered} \tag{78}$$

Unfortunately, the operator P does not appear to have any simple structure. If it could be simplified, then one might hope to be able to establish the expected generalization of these results for the eight-vertex model.

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